

# FINITELY CONVERGENT ALGORITHM FOR NONCONVEX INEQUALITY PROBLEMS

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ABSTRACT. We extend Fukushima's result on the finite convergence of an algorithm for the global convex feasibility problem to the local nonconvex case.

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## 1. INTRODUCTION

Let  $X$  be a Hilbert space. We consider the *Nonconvex Inequality Problem* (NIP)

$$(NIP): \quad \text{For } f : X \rightarrow \mathbb{R}, \text{ find } x \in \mathbb{R}^n \text{ s.t. } f(x) \leq 0. \quad (1.1)$$

In [Fuk82], Fukushima proposed a simple *global* algorithm when  $X = \mathbb{R}^n$  for the Convex Inequality Problem (which is the NIP with the additional requirement that  $f(\cdot)$  is convex) that converges to some point  $\bar{x}$  such that  $f(\bar{x}) \leq 0$  if the Slater condition (i.e., the existence of a point  $x^*$  satisfying  $f(x^*) < 0$ ) is satisfied. The ideas can be easily extended to the case when  $X$  is a Hilbert space, and the function  $f(\cdot)$  need not be smooth. In this paper, we make use of tools in nonsmooth and variational analysis [Cla83, Mor06, RW98] to prove a local result on the case where  $f(\cdot)$  is nonconvex.

We now discuss some problems related to the NIP. In the case where  $f(\cdot)$  can be written as a maximum of finitely many smooth functions, a variant of the Newton method converges superlinearly, and global convergence is possible when  $f(\cdot)$  is the maximum of finitely many smooth convex functions. We refer to the references stated in [Fuk82] for more details. (It appears that [PI88] have obtained similar results independently.)

In [Rob76], Robinson considered the *K-Convex Inequality Problem* (KCIP), which is a generalization of the (CIP). For  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , and a closed convex

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cone  $K \subset \mathbb{R}^m$ , we write  $y_1 \leq_K y_2$  if  $y_2 - y_1 \in K$ . The KCIP is defined by

$$(\text{KCIP}): \text{ For } f : \mathbb{R}^n \rightarrow \mathbb{R}^m \text{ and } C \subset \mathbb{R}^n, \text{ find } x \in C \text{ s.t. } f(x) \leq_K 0. \quad (1.2)$$

Robinson's algorithm in [Rob76] for the CIP can be described as follows: At each iterate  $x_i$ , a subgradient  $y_i \in \partial f(x_i)$  is obtained, and the halfspace

$$H_i^< := \{x \in \mathbb{R}^n \mid f(x_i) + \langle y_i, x - x_i \rangle \leq 0\} \quad (1.3)$$

contains  $f^{-1}((-\infty, 0])$ . The next iterate  $x_{i+1}$  is obtained by projecting  $x_i$  onto  $H_i^<$ . Assuming regularity and convexity (and no smoothness), Robinson proved that the algorithm for the KCIP converges at least linearly. With smoothness, superlinear convergence can be expected.

Modifications for a finitely convergent algorithm for the NIP can be traced back to [PM79, MPH81], where  $f(\cdot)$  is a maximum of finitely many smooth functions. The main idea for obtaining finite convergence under the Slater condition can be described as follows. Instead of trying to find  $x$  such that  $f(x) \leq 0$ , an infinite sequence  $\{\epsilon_i\}$  of positive numbers is introduced, and one tries to find  $x$  satisfying

$$f(x) \leq -\epsilon_i$$

in the  $i$ th iteration. The contribution in [Fuk82] is to show that the smoothness conditions can be dropped. For more recent work, we refer the reader to [BWWX14, CCP11, Cro04] and the references therein.

A problem related to the NIP is the Set Intersection Problem (SIP). For sets  $K_1, \dots, K_r$  in a Hilbert space  $X$ , the SIP is stated as:

$$(\text{SIP}): \text{ Find } x \in K := \bigcap_{i=1}^r K_i, \text{ where } K \neq \emptyset. \quad (1.4)$$

The SIP can be seen as a particular case of the NIP: Take  $f(\cdot)$  to be  $\max_{i=1, \dots, r} d(x, K_i)$ . A common method of solving such problems is the method of alternating projections, which typically has linear convergence even in convex problems. There has been recent interest in nonconvex problems [LM08, LLM09], where the research is focused on conditions for the linear convergence of the method of alternating projections and its variants.

We also remark that the NIP is related to filter methods for nonlinear programming [FL02].

**1.1. Contributions of this paper.** In this paper, we prove a local result on the finite convergence of an algorithm for the NIP (1.1) when  $f(\cdot)$  is approximately convex [NLT00, DG04] (See Definition 2.2 and the subsequent commentary) and  $X$  is a Hilbert space.

**1.2. Notation.** Let  $X$  be a Hilbert space, and let  $x \in X$  and  $S \subset X$ . The following notation we will use is quite standard.

$\mathbb{B}(x, r)$       The closed ball with center  $x$  and radius  $r$ .  
 $d(x, S)$       The distance from  $x$  to  $S$ .

## 2. PRELIMINARIES

In this section, we provide the necessary background in variational analysis for the proof of our algorithm for the NIP. We first recall the Clarke subdifferential.

**Definition 2.1.** (Clarke subdifferential) Let  $X$  be a Hilbert space. Consider a function  $f : X \rightarrow \mathbb{R}$  locally Lipschitz at a point  $\bar{x} \in X$ . The *Clarke (generalized) subdifferential* of  $f$  at  $\bar{x} \in X$  is defined by

$$\partial f(\bar{x}) := \{x^* \in X^* : \langle x^*, d \rangle \leq f^0(\bar{x}; d) \text{ for all } d \in X\}, \quad (2.1)$$

where  $f^0(\cdot; \cdot)$  is the *Clarke (generalized) directional derivative* defined by

$$f^0(\bar{x}; d) := \limsup_{(y,t) \rightarrow (\bar{x}, 0^+)} \frac{f(y + td) - f(y)}{t}. \quad (2.2)$$

We now describe the nonconvex functions for which we are able to prove finite convergence of our algorithm.

**Definition 2.2.** [NLT00] (Approximate convexity) Let  $X$  be a Hilbert space and  $f : X \rightarrow \mathbb{R}$ . We say that  $f(\cdot)$  is *approximately convex* at  $\bar{x}$  if for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$f(y) \geq f(x) + \langle s, y - x \rangle - \epsilon \|y - x\| \text{ for all } x, y \in \mathbb{B}(\bar{x}, \delta) \text{ and } s \in \partial f(x).$$

The notion of weak convexity in [Via83] (see also [HU84] and the references therein) was a precursor to the notion of approximate convexity in [NLT00]. The definition of approximate convexity above is different from its usual definition, but is equivalent by [DG04, Theorem 1]. In the case where  $X = \mathbb{R}^n$ , approximate convexity is equivalent to  $f(\cdot)$  being lower- $\mathcal{C}^1$  [DG04, Spi81, ADT04]. Lower- $\mathcal{C}^1$  functions include the pointwise maximum of a finite number of  $\mathcal{C}^1$  functions. We refer to [RW98, Section 10G] and the references therein for a discussion on lower- $\mathcal{C}^1$  functions, and more generally, subsmooth functions.

We now recall metric regularity.

**Definition 2.3.** (Metric regularity) Let  $S : X \rightrightarrows Y$  be a set-valued map. We say that  $S(\cdot)$  is *metrically regular* at  $(\bar{x}, \bar{y})$  if there exist a constant  $\kappa \geq 0$  and neighborhoods  $U$  of  $\bar{x}$  and  $V$  of  $\bar{y}$  such that

$$d(x, S^{-1}(u)) \leq \kappa d(u, S(x)) \text{ for all } x \in U \text{ and } u \in V.$$

We now make a claim about locally Lipschitz functions.

**Proposition 2.4.** (Metric regularity of epigraphical maps) Let  $X$  be a Hilbert space, and let  $f : X \rightarrow \mathbb{R}$  be locally Lipschitz at  $\bar{x} \in X$ . If  $0 \notin \partial f(\bar{x})$ , then the epigraphical map  $E : X \rightrightarrows \mathbb{R}$  defined by  $E(x) := [f(x), \infty)$  is metrically regular at  $(\bar{x}, f(\bar{x}))$ .

*Proof.* We make use of the Aubin criterion in [DQZ06, Theorem 1.2], but we need to recall a few definitions. Let  $X$  and  $Y$  be Banach spaces. For a positively homogeneous map  $H : X \rightrightarrows Y$ , the *inner norm*  $\|\cdot\|^-$  is defined as

$$\|H\|^- := \sup_{x \in \mathbb{B}} \inf_{y \in H(x)} \|y\|.$$

For a set-valued map  $S : X \rightrightarrows Y$ , consider  $(\bar{x}, \bar{y})$  such that  $\bar{y} \in S(\bar{x})$ , or  $(\bar{x}, \bar{y}) \in \text{Graph}(S)$ . The *graphical (contingent) derivative* of  $S$  at  $(\bar{x}, \bar{y})$  is defined by

$$\text{Graph}(DS(\bar{x} \mid \bar{y})) := T_{\text{Graph}(S)}(\bar{x}, \bar{y}),$$

where the tangent cone  $T_{\text{Graph}(S)}(\bar{x}, \bar{y})$  is defined as follows:  $(u, v) \in T_{\text{Graph}(S)}(\bar{x}, \bar{y})$  if and only if there exists sequences  $t_n \searrow 0$ ,  $u_n \rightarrow u$  and  $v_n \rightarrow v$  such that  $\bar{y} + t_n v_n \in S(\bar{x} + t_n u_n)$ .

The Aubin criterion states that for Banach spaces  $X$  and  $Y$  and a set-valued map  $S : X \rightrightarrows Y$ ,  $S(\cdot)$  is metrically regular at  $(\bar{x}, \bar{y})$  if

$$\limsup_{\substack{(x,y) \rightarrow (\bar{x}, \bar{y}) \\ (x,y) \in \text{Graph}(S)}} \|DS(x | y)^{-1}\|^- \quad (2.3)$$

is finite.

We now apply the Aubin criterion to our particular setting. Since  $0 \notin \partial f(\bar{x})$ , by the formulation of the Clarke subdifferential using the Clarke directional derivative (2.2), there exists a direction  $d$ , where  $\|d\| = 1$ , such that  $f^0(\bar{x}, d) < -\mu$ , where  $\mu > 0$ . This means that if  $(x, t)$  are close enough to  $(\bar{x}, 0^+)$ , then

$$\frac{f(x + td) - f(x)}{t} < -\mu.$$

This in turn implies that  $(d, -\mu) \in \text{Graph}(DE(x | f(x)))$ . In other words,

$$\frac{1}{\mu}d \in DE(x | f(x))^{-1}(-1). \quad (2.4)$$

Since  $(0, 1)$  is a recession direction in  $\text{Graph}(S)$ , it is clear that

$$0 \in DE(x | f(x))^{-1}(1). \quad (2.5)$$

Whenever  $y > f(x)$  and  $x$  is close enough to  $\bar{x}$ , the local Lipschitz continuity of  $f(\cdot)$  at  $\bar{x}$  ensures that  $(x, y)$  is in the interior of the epigraph of  $f$ , from which we get

$$0 \in DE(x | y)^{-1}(1) \text{ and } 0 \in DE(x | y)^{-1}(-1) \text{ whenever } y > f(x). \quad (2.6)$$

The formulas (2.4), (2.5) and (2.6) combine to give us  $\|DE(x | y)^{-1}\|^- \leq 1/\mu$  for all  $(x, y) \in \text{Graph}(E)$  close enough to  $(\bar{x}, f(\bar{x}))$ . Thus the Aubin criterion applies to give us the metric regularity of  $E(\cdot)$  at  $(\bar{x}, f(\bar{x}))$ .  $\square$

### 3. FINITELY CONVERGENT ALGORITHM FOR THE NIP

We now present our algorithm for the NIP, and prove its finite convergence in Theorem 3.3.

**Algorithm 3.1.** (*Finitely convergent algorithm for NIP*) Let  $X$  be a Hilbert space. Consider a function  $f : X \rightarrow \mathbb{R}$ , a point  $x_0$  and a sequence  $\{\epsilon_i\}$  of strictly decreasing positive numbers converging to zero. This algorithm seeks to find a point  $x'$  such that  $f(x') < 0$ .

**Step 0:** Set  $i = 0$ .

**Step 1:** Find  $s_i^{(j)} \in \partial f(x_i)$  for  $j = 1, \dots, J_i$ , where  $J_i$  is some finite number. Let  $x_i^{(j)}$  be  $x_i - \frac{\epsilon_i + f(x_i)}{\|s_i^{(j)}\|^2} s_i^{(j)}$ , which is also the projection of  $x_i$  onto the set  $\{x : f(x_i) + \langle s_i^{(j)}, x - x_i \rangle \leq -\epsilon_i\}$ . Consider the polyhedron

$$P_i := \{x : f(x_i) + \langle s_i^{(j)}, x - x_i \rangle \leq -\epsilon_i \text{ for all } j \in \{1, \dots, J_i\}\}.$$

The next iterate  $x_{i+1}$  is obtained by projecting  $x_i$  onto  $P_i$ .

**Step 2:** Increase  $i$  and go back to step 1 till convergence.

Before proving Theorem 3.3, we recall a simple principle that will be used in the proof there.

**Proposition 3.2.** (*Projections onto polyhedra*) Let  $F$  be a polyhedron in a Hilbert space  $X$  such that

$$F := \bigcap_{i=1}^k \{x : \langle x, a_i \rangle \leq b_i\},$$

where  $a_i \in X$  and  $b_i \in \mathbb{R}$  for  $i \in \{1, \dots, k\}$ . Choose a point  $x_0$  and let  $x_1 := P_F(x_0)$ . Then for any  $y \in F$ , we have  $\langle x_0 - x_1, y - x_1 \rangle \leq 0$ .

We now prove that Algorithm 3.1 can converge in finitely many iterations to such a point  $x'$ . Our proof is an extension of the proof in [Fuk82].

**Theorem 3.3.** (*Finite convergence of Algorithm 3.1*) Let  $X$  be a Hilbert space. Consider a locally Lipschitz function  $f : X \rightarrow \mathbb{R}$ . Let  $\bar{x}$  be such that

- (1)  $f(\bar{x}) = 0$ ,
- (2)  $0 \notin \partial f(\bar{x})$ , and
- (3)  $f(\cdot)$  is approximately convex at  $\bar{x}$ .

Suppose also that the strictly decreasing sequence  $\{\epsilon_i\}_i$  converges to zero at a sub-linear rate (i.e., slower than any linearly convergent sequence). There is a neighborhood  $U$  of  $\bar{x}$  and a number  $\bar{\epsilon}$  such that if  $x_0 \in U$  and  $\epsilon_0 < \bar{\epsilon}$ , then Algorithm 3.1 converges in finitely many iterations. (i.e.,  $f(x_i) \leq 0$  for some  $i$ .)

*Proof.* Seeking a contradiction, we assume  $f(x_i) > 0$  for all  $i$ . Our proof is broken up into several parts.

**Claim 1:** There is a neighborhood  $U$  of  $\bar{x}$  and  $\bar{\epsilon} > 0$  such that if  $\tilde{x}_1 \in U$  and  $f(\tilde{x}_1) > 0$ , then for any  $\tilde{\epsilon} \in (0, \bar{\epsilon}]$  and  $s^{(j)} \in \partial f(\tilde{x}_1)$ , where  $j \in \{1, \dots, J\}$ , the projection of  $\tilde{x}_1$  onto the polyhedron

$$P := \{x : f(\tilde{x}_1) + \langle s^{(j)}, x - \tilde{x}_1 \rangle \leq -\tilde{\epsilon} \text{ for all } j \in \{1, \dots, J\}\} \quad (3.1)$$

lies in  $U$ .

By the Clarke directional derivative (2.2) of  $f(\cdot)$  at  $\bar{x}$ , since  $0 \notin \partial f(\bar{x})$ , there exists a direction  $d$ , where  $\|d\| = 1$ , and  $\mu > 0$  such that

$$\limsup_{t \rightarrow 0} \frac{1}{t} [f(\bar{x} + td) - f(\bar{x})] < -\mu.$$

In particular, this implies that if  $\bar{t}$  is small enough, then  $f(\bar{x} + td) < f(\bar{x}) - \mu t = -\mu t$  for all  $t \in [0, \bar{t}]$ . Then by the approximate convexity of  $f(\cdot)$  at  $\bar{x}$ , for any  $\epsilon_{ac} > 0$ , there is a neighborhood  $U_1$  of  $\bar{x}$  such that

$$f(y) \geq f(x) + \langle s^{(j)}, y - x \rangle - \epsilon_{ac} \|y - x\| \text{ for all } x, y \in U_1 \text{ and } j \in \{1, \dots, J\}.$$

To simplify our notation, we let  $S_\epsilon := f^{-1}((-\infty, -\epsilon])$  just like in [Fuk82]. Recall  $E(\cdot)$ , the epigraphical map of  $f(\cdot)$  defined in Proposition 2.4, is metrically regular at  $\bar{x}$ . This means that by lowering  $\bar{\epsilon}$  if necessary, there is a  $\kappa \in [\bar{\kappa}, \bar{\kappa} + 1]$ , a  $\delta > 0$  and a neighborhood  $U_2$  of  $\bar{x}$  such that if  $x \in U_2$  and  $\epsilon < \bar{\epsilon}$ , then

$$d(x, S_\epsilon) = d(x, E^{-1}(-\epsilon)) \leq \kappa d(E(x), -\epsilon) = \kappa [f(x) + \epsilon]. \quad (3.2)$$

We impose the following requirements on  $\epsilon_{ac}$ ,  $\bar{t}$  and  $\bar{\epsilon}$ .

- (R1) Let  $\epsilon_{ac} > 0$  be small enough so that  $2\epsilon_{ac} < \mu$  and  $(\bar{\kappa} + 1)\epsilon_{ac} < \frac{1}{3}$ .
- (R2) Let  $\bar{t} > 0$  be small enough so that

$$\begin{aligned} \mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t}) &\subset U_1 \cap U_2, \\ \text{and } L := \sup_{s \in \partial f(\mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t}))} \|s\| &\text{ is finite.} \end{aligned}$$

(R3) Let  $\bar{\epsilon} > 0$  be small enough so that  $\bar{\epsilon} + 2\epsilon_{ac}\bar{t} < \bar{t}\mu$ .

(R4) Reduce  $\bar{t}$  and  $\bar{\epsilon}$  if necessary so that (R2) and (R3) holds, and

$$\mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t} + [\bar{\kappa} + 1][3L\bar{t} + \bar{\epsilon}]) \subset U_1.$$

The finiteness of  $L$  in (R2) is possible for some  $\bar{t} > 0$  by making use of [Cla83, Proposition 2.1.2(a)] and the fact that  $f$  is locally Lipschitz at  $\bar{x}$ . We can now prove Claim 1 for  $U = \mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t})$ . We will only need (R1)-(R3) for now, and the significance of (R4) will be explained in Claim 2. Consider any  $\tilde{x}_1 \in \mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t})$  such that  $f(\tilde{x}_1) > 0$ . For any  $s \in \partial f(\tilde{x}_1)$ , we have

$$\begin{aligned} & f(\tilde{x}_1) + \left\langle s, \frac{-[\tilde{\epsilon} + f(\tilde{x}_1)]}{\|s\|^2} s \right\rangle \\ &= -\tilde{\epsilon} \\ &> -\bar{t}\mu + \epsilon_{ac}2\bar{t} \text{ (Using (R3) and the fact that } \tilde{\epsilon} \leq \bar{\epsilon}.) \\ &> f(\bar{x} + \bar{t}d) + \epsilon_{ac}2\bar{t} \\ &\geq f(\tilde{x}_1) + \langle s, (\bar{x} + \bar{t}d) - \tilde{x}_1 \rangle - \epsilon_{ac}\|[\bar{x} + \bar{t}d] - \tilde{x}_1\| + \epsilon_{ac}2\bar{t} \\ &\geq f(\tilde{x}_1) + \langle s, (\bar{x} + \bar{t}d) - \tilde{x}_1 \rangle \text{ (since } \tilde{x}_1 \in \mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t})). \end{aligned}$$

This implies that  $\left\langle s^{(j)}, \frac{-[\tilde{\epsilon} + f(\tilde{x}_1)]}{\|s^{(j)}\|^2} s^{(j)} + \tilde{x}_1 - [\bar{x} + \bar{t}d] \right\rangle \geq 0$  for all  $j \in \{1, \dots, J\}$ .

Let  $\tilde{v}^{(j)} = \frac{[\tilde{\epsilon} + f(\tilde{x}_1)]}{\|s^{(j)}\|^2} s^{(j)}$ . We have

$$\langle \tilde{x}_1 - [\tilde{x}_1 - \tilde{v}^{(j)}], [\bar{x} + \bar{t}d] - [\tilde{x}_1 - \tilde{v}^{(j)}] \rangle \leq 0. \quad (3.3)$$

In other words, the angle  $\angle \tilde{x}_1[\tilde{x}_1 - \tilde{v}^{(j)}][\bar{x} + \bar{t}d] \geq \pi/2$ .

The polyhedron  $P$  in (3.1) can also be written as

$$P = \{x : \langle x - [\tilde{x}_1 - \tilde{v}^{(j)}], s^{(j)} \rangle \leq 0 \text{ for all } j \in \{1, \dots, J\}\}.$$

In view of (3.3) and the above discussion, the point  $\bar{x} + \bar{t}d$  lies in  $P$ . The projection of  $\tilde{x}_1$  onto  $P$ , say  $\tilde{x}_2$ , creates a hyperplane that separates  $\tilde{x}_1$  and  $\bar{x} + \bar{t}d$ . In other words,  $\angle \tilde{x}_1\tilde{x}_2[\bar{x} + \bar{t}d] \geq \pi/2$ . This in turn implies that we have  $\|\tilde{x}_2 - [\bar{x} + \bar{t}d]\| \leq \|\tilde{x}_1 - [\bar{x} + \bar{t}d]\|$ . In other words,  $\tilde{x}_1 \in \mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t})$  implies  $\tilde{x}_2 \in \mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t})$ . This ends the proof of Claim 1 with  $U = \mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t})$ .

It is easy to see that this implies that if  $x_0 \in \mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t})$ , then the iterates  $x_i$  generated by Algorithm 3.1 lie in  $\mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t})$  as well, provided the starting  $\epsilon_0$  is smaller than  $\bar{\epsilon}$ .

**Claim 2:** Let  $p_i := P_{S_{\epsilon_i}}(x_i)$ , the projection of  $x_i$  onto  $S_{\epsilon_i}$ . If  $x_i \in \mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t})$ , then  $p_i$  lies in  $U_1$ .

From (3.2), we have

$$\|p_i - x_i\| = d(x_i, S_{\epsilon_i}) \leq \kappa[f(x_i) + \epsilon_i] \leq [\bar{\kappa} + 1][f(x_i) + \epsilon_i]. \quad (3.4)$$

Since  $x_i \in \mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t})$ , we have  $\|x_i - \bar{x}\| \leq 3\bar{t}$ . It is well known that the constant  $L$  in (R2) is also an upper bound on the Lipschitz constant of  $f$  in  $\mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t})$  (for example, through the Mean Value Theorem in [Cla83, Theorem 2.3.7] or [Leb75]), so  $f(x_i)$  is bounded from above by  $3L\bar{t}$ . Hence

$$\|p_i - (\bar{x} + \bar{t}d)\| \leq \|x_i - (\bar{x} + \bar{t}d)\| + \|p_i - x_i\| \leq 2\bar{t} + [\bar{\kappa} + 1][3L\bar{t} + \bar{\epsilon}].$$

By (R4), we can see that  $p_i \in U_1$  as needed. This ends the proof of Claim 2.

**Claim 3:** The sequence  $\{d(x_i, S_{\epsilon_i})\}_i$  converges at least linearly to 0.

From the continuity of  $f(\cdot)$ , it is clear that  $f(p_i) = -\epsilon_i$ . For the choice  $s_i^{(j)} \in \partial f(x_i)$ , we recall that  $x_i, p_i \in U_1$ , and get

$$\begin{aligned} f(p_i) &\geq f(x_i) + \langle s_i^{(j)}, p_i - x_i \rangle - \epsilon_{ac} \|p_i - x_i\| \\ \langle s_i^{(j)}, p_i - x_i \rangle &\leq f(p_i) - f(x_i) + \epsilon_{ac} \|p_i - x_i\| \\ &= -\epsilon_i - f(x_i) + \epsilon_{ac} \|p_i - x_i\|. \end{aligned}$$

Recall  $x_i^{(j)} = x_i - \frac{[\epsilon_i + f(x_i)]}{\|s_i^{(j)}\|^2} s_i^{(j)}$ . Let

$$\tilde{x}_i^{(j)} := \frac{2}{3} x_i^{(j)} + \frac{1}{3} x_i. \quad (3.5)$$

It is easy to check that  $\langle s_i^{(j)}, x_i^{(j)} - x_i \rangle = -\epsilon_i - f(x_i)$  and  $\langle s_i^{(j)}, \tilde{x}_i^{(j)} - x_i \rangle = -\frac{2}{3}[\epsilon_i + f(x_i)]$ .

From (3.4) and the preceding discussion, we have

$$\begin{aligned} \frac{\langle s_i^{(j)}, p_i - x_i \rangle}{\langle s_i^{(j)}, \tilde{x}_i^{(j)} - x_i \rangle} &\geq \frac{[\epsilon_i + f(x_i)] - \epsilon_{ac} \|x_i - p_i\|}{\frac{2}{3}[\epsilon_i + f(x_i)]} \\ &\geq \frac{\|x_i - p_i\|/[\bar{\kappa} + 1] - \epsilon_{ac} \|x_i - p_i\|}{\frac{2}{3}\|x_i - p_i\|/[\bar{\kappa} + 1]} \\ &= \frac{3}{2}[1 - [\bar{\kappa} + 1]\epsilon_{ac}]. \end{aligned} \quad (3.6)$$

In view of  $[\bar{\kappa} + 1]\epsilon_{ac} < 1/3$  in (R1), the ratio  $\frac{3}{2}[1 - [\bar{\kappa} + 1]\epsilon_{ac}]$  is greater than 1. Since  $\tilde{x}_i^{(j)} - x_i$  is in the direction of  $-s_i^{(j)}$ , the angle  $\angle p_i \tilde{x}_i^{(j)} x_i$  is greater than  $\pi/2$ . (See Figure 3.1.) In other words, the point  $p_i$  is in the polyhedron

$$\tilde{P}_i := \{x : \langle x - \tilde{x}_i^{(j)}, s_i^{(j)} \rangle \leq 0 \text{ for all } j \in \{1, \dots, J_i\}\}.$$

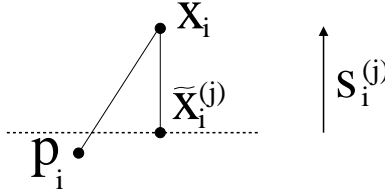


FIGURE 3.1. This figure explains the setup in (3.6) and why  $\angle p_i \tilde{x}_i^{(j)} x_i > \pi/2$ .

Let the projection of  $x_i$  onto  $\tilde{P}_i$  be  $\tilde{x}_{i+1}$ . It is clear to see that  $\tilde{P}_i$  is the polyhedron created by scaling  $P_i$  about  $x_i$  with a factor of  $2/3$  by (3.5). Thus  $\tilde{x}_{i+1} = \frac{2}{3}x_{i+1} + \frac{1}{3}x_i$ . We can also infer (using the principle in Proposition 3.2) that

$$\angle x_i \tilde{x}_{i+1} p_i \geq \pi/2. \quad (3.7)$$

Let  $\tilde{p}_i$  be the projection of  $p_i$  onto the line connecting  $x_{i+1}$  and  $x_i$ . We can infer from (3.7) that  $\tilde{x}_{i+1}$  must lie between  $\tilde{p}_i$  and  $x_i$ . Thus

$$\begin{aligned} \|x_{i+1} - p_i\|^2 &= \|x_i - p_i\|^2 - \|\tilde{p}_i - x_i\|^2 + \|\tilde{p}_i - x_{i+1}\|^2 \\ &\leq \|x_i - p_i\|^2 - \|\tilde{x}_{i+1} - x_i\|^2 + \|\tilde{x}_{i+1} - x_{i+1}\|^2 \\ &= \|x_i - p_i\|^2 - \frac{4}{9}\|x_{i+1} - x_i\|^2 + \frac{1}{9}\|x_{i+1} - x_i\|^2 \\ &= \|x_i - p_i\|^2 - \frac{1}{3}\|x_{i+1} - x_i\|^2. \end{aligned} \quad (3.8)$$

We now bound the distance  $\|x_{i+1} - x_i\|$ . From (3.2), we have

$$\|p_i - x_i\| \leq \kappa[f(x_i) + \epsilon_i] = \kappa \left\| \frac{[f(x_i) + \epsilon_i]}{\|s_i^{(1)}\|^2} s_i^{(1)} \right\| \|s_i^{(1)}\| \leq \kappa \|x_i - x_i^{(1)}\| \sup_{s' \in \partial f(x_i)} \|s'\|.$$

We let  $r := 1/[\kappa \sup_{s' \in \partial f(\mathbb{B}(\bar{x} + \bar{t}d, 2\bar{t}))} \|s'\|]$ . Then we have

$$\|x_i - x_i^{(1)}\| \geq r \|p_i - x_i\|. \quad (3.9)$$

Since  $x_i^{(1)}$  is the projection of  $x_i$  onto the halfspace

$$H_i^{(1)} := \{x : \langle x - x_i^{(1)}, s_i^{(1)} \rangle \leq 0\},$$

$x_{i+1}$  is the projection of  $x_i$  onto  $P_i$ , and  $P_i \subset H_i^{(1)}$ , we must have  $\|x_{i+1} - x_i\| \geq \|x_i - x_i^{(1)}\|$ . Combining with (3.8) and (3.9), we have

$$\begin{aligned} \|x_{i+1} - p_i\|^2 &\leq \|x_i - p_i\|^2 - \frac{1}{3}\|x_{i+1} - x_i\|^2 \\ &\leq \|x_i - p_i\|^2 - \frac{1}{3}r^2\|x_i - p_i\|^2 = \left[1 - \frac{1}{3}r^2\right] \|x_i - p_i\|^2. \end{aligned}$$

Hence

$$d(x_{i+1}, S_{\epsilon_{i+1}}) \leq d(x_{i+1}, S_{\epsilon_i}) \leq \sqrt{1 - \frac{1}{3}r^2} \|x_i - p_i\| = \sqrt{1 - \frac{1}{3}r^2} d(x_i, S_{\epsilon_i}),$$

which gives at least a linear rate of decrease of  $\{d(x_i, S_{\epsilon_i})\}_i$ . This ends the proof of Claim 3.

**Claim 4:** The sequence  $\{d(x_i, S_{\epsilon_i})\}_i$  converges to 0 at a sublinear rate, contradicting Claim 3.

Recall  $p_i = P_{S_{\epsilon_i}}(x_i)$ . We have

$$\begin{aligned} -\epsilon_i &= f(p_i) \\ &\geq f(x_i) + \langle s_i, p_i - x_i \rangle - \epsilon_{ac} \|p_i - x_i\| \\ &\geq \langle s_i, p_i - x_i \rangle - \epsilon_{ac} \|p_i - x_i\| \\ &\geq -[\|s_i\| + \epsilon_{ac}] \|p_i - x_i\|, \end{aligned}$$

which gives  $\|x_i - p_i\| \geq \frac{\epsilon_i}{\|s_i\| + \epsilon_{ac}}$ , so

$$d(x_i, S_{\epsilon_i}) = \|x_i - p_i\| \geq \frac{\epsilon_i}{\|s_i\| + \epsilon_{ac}} \geq \frac{\epsilon_i}{L + \epsilon_{ac}}.$$

This implies that the sequence  $\{d(x_i, S_{\epsilon_i})\}$  converges at a sublinear rate, which contradicts Claim 3. This ends the proof of Claim 4.

Thus, the sequence  $\{x_i\}$  has to terminate finitely.  $\square$



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## REFERENCES

- [ADT04] D. Aussel, A. Daniilidis, and L. Thibault, *Subsmooth sets: Functional characterizations and related concepts*, Trans. Amer. Math. Soc. **357** (2004), no. 4, 1275–1301.
- [BWWX14] H.H. Bauschke, Caifang Wang, Xianfu Wang, and Jia Xu, *On the finite convergence of a projected cutter method*, ArXiv e-prints (2014).
- [CCP11] Y. Censor, W. Chen, and H. Pajoohesh, *Finite convergence of a subgradient projection method with expanding controls*, Applied Mathematics and Optimization **64** (2011), no. 2, 273–285.
- [Cla83] F.H. Clarke, *Optimization and nonsmooth analysis*, Wiley, Philadelphia, 1983, Republished as a SIAM Classic in Applied Mathematics, 1990.
- [Cro04] G. Crombez, *Finding common fixed points of a class of paracontractions*, Acta Mathematica Hungarica **103** (2004), no. 3, 233–241.
- [DG04] A. Daniilidis and P. Georgiev, *Approximate convexity and submonotonicity*, J. Math. Anal. Appl. **291** (2004), 292–301.
- [DQZ06] A.L. Dontchev, M. Quincampoix, and N. Zlateva, *Aubin criterion for metric regularity*, J. Convex Anal. **3** (2006), 45–63.
- [FL02] R. Fletcher and S. Leyffer, *Nonlinear programming without a penalty function*, Math. Program. **91** (2002), no. 2, 239–269.
- [Fuk82] M. Fukushima, *A finitely convergent algorithm for convex inequalities*, IEEE Trans. Automat. Control **27** (1982), no. 5, 1126–1127.
- [HU84] J.-B. Hiriart-Urruty, *Generalized differentiability, duality and optimization for problems dealing with differences of convex functions*, Lecture Notes in Econom. Math. Systems, vol. 256, 1984, pp. 37–70.
- [Leb75] G. Lebourg, *Valeur moyenne pour gradient généralisé*, C. R. Acad. Sci. Paris **281** (1975), 795–797.
- [LLM09] A.S. Lewis, D.R. Luke, and J. Malick, *Local linear convergence for alternating and averaged nonconvex projections*, Found. Comput. Math. **9** (2009), no. 4, 485–513.
- [LM08] A.S. Lewis and J. Malick, *Alternating projection on manifolds*, Math. Oper. Res. **33** (2008), 216–234.
- [Mor06] B.S. Mordukhovich, *Variational analysis and generalized differentiation I and II*, Springer, Berlin, 2006, Grundlehren der mathematischen Wissenschaften, Vols 330 and 331.
- [MPH81] D.Q. Mayne, E. Polak, and A.J. Heunis, *Solving nonlinear inequalities in a finite number of iterations*, J. Optim. Theory Appl. **33** (1981), 207–221.
- [NLT00] H. Van Ngai, D. T. Luc, and M. Théra, *Approximate convex functions*, J. Nonlinear Convex Anal. **1** (2000), no. 2, 155–176.
- [PI88] A.R. De Pierro and A.N. Iusem, *A finitely convergent "row-action" method for the convex feasibility problem*, Appl. Math. Optim. **17** (1988), 225–235.
- [PM79] E. Polak and D.Q. Mayne, *On the finite solution of nonlinear inequalities*, IEEE Trans. Automat. Control **AC-24** (1979), 443–445.
- [Rob76] S.M. Robinson, *A subgradient algorithm for solving  $K$ -convex inequalities*, Optimization and operations research (Proc. Conf., Oberwolfach, 1975), Lecture Notes in Econom. Math. Systems, Vol. 117 (W Oettli and K. Ritter, eds.), Springer, Berlin-New York, 1976, pp. 237–245.
- [RW98] R.T. Rockafellar and R.J.-B. Wets, *Variational analysis*, Grundlehren der mathematischen Wissenschaften, vol. 317, Springer, Berlin, 1998.
- [Spi81] J.E. Spingarn, *Submonotone subdifferentials of Lipschitz functions*, Trans. Amer. Math. Soc. **264** (1981), 77–89.
- [Via83] J.-P. Vial, *Strong and weak convexity of sets and functions*, Math. Oper. Res. **8** (1983), 231–259.

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